Chain Ladder Reserving Methods for Liabilities with Per Occurrence Limits

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Abstract

Motivation: As an insurance regulator, I regularly see instances where maximum limit losses are removed from incurred and/or paid losses prior to application of the development factors. In some of these instances, the triangles and LDFs are created with limited losses, as opposed to unlimited losses.

Method: This paper simulates loss development triangles that include maximum limit losses. It compares exclusion vs. inclusion of maximum limit losses to show how each option affects the accuracy of the results. This paper provides simulated empirical probabilities obtained by randomly dispersing large losses throughout a triangle, then calculating the ultimate limited losses by two different methods.

Conclusion: If limited LDFs are calculated using triangles that include truncated maximum limit losses, then excluding maximum limit losses prior to application of the LDF produces an understated ultimate and reserve.

Availability. Calculations were performed using @RISK Standard version 5.0, from Palisade Corporation, Ithaca, NY, U.S.A. The commercial software package @Risk was used to simulate loss triangles and to create graphs of empirical loss distributions. The Excel/@Risk spreadsheets used for calculating triangles with randomly disbursed large losses are available through the author.

Keywords. Loss development; reserving, data organization, net reserves, gross reserves, ceded reserves, reserving methods, aggregate excess/stop loss; simulation

1. INTRODUCTION

It is not uncommon to see a reserve analysis in which the actuary has removed full limit losses from paid or incurred data prior to application of loss development factors. (The full limit losses are added back in after application of the LDFs) This paper provides examples showing that if used improperly, this commonly used technique understates reserves. If the LDFs are estimated using all losses, including truncated losses, and the LDFs are applied only to the losses below the limits, then the reserve is under-estimated. This is due to the fact that losses reaching the limits no longer develop over time and hence the LDFs estimated using all losses are smaller than the LDFs estimated using only the losses below the limits.

1.1 Research Context

The focus area addressed is reserving methods applicable to data limited to a certain per occurrence limit.

[1] Jennifer Wu, an actuary at the Texas Department of Insurance went above and beyond the call of duty as a reviewer of this paper.

There are several papers that discuss issues tangentially related to the one discussed here. For example, Daley [3] and Klemmt [5] discuss potential increases in accuracy gained by applying methods differently to large losses vs. small losses. Several papers such as Brown [1], Halliwell [4] and Pinto [6] discuss using and or calculating percentages of losses within various layers. However, I was unable to find any papers focusing on the issue addressed by this particular paper i.e. removal of large losses prior to application of the LDF, but where the LDFs were calculated with the truncated losses included. It is possible that no one has written such a paper because the conclusion appeared to be obvious. Nevertheless, the technique is used[2], so consequently I am writing this paper.

1.2 Objective

The objective of this paper is to increase awareness within the actuarial community that application of a commonly used technique is actuarially unsound.

1.3 Outline

The remainder of the paper will provide an example and some simulation results showing that it is more accurate to apply the limited LDFs to all the losses rather than to only the losses that are below the limit. The paper will provide some discussion about why intuitively these results make sense.

2. BACKGROUND AND METHODS

2.1 Background – Applying the LDF to the Losses

Suppose that you are given the following information

Total Case Inc. Limited Losses:	$3M
Insured Limit:	$500K
Losses exceeding 100K:	120K, 450K, 500K
Applicable Incurred LDF:	1.2

Note that I did not explain how the incurred LDF was calculated. This is an important piece of information. However, for now, let us suppose that you do not know how the LDF was calculated.

[2] One reviewer of the proposal exclaimed, "Make them stop!"

With the information at hand there are a couple of different ways to proceed.

2.1.1 Method-A

We could multiply $3M by 1.2 to obtain $3.6M as the ultimate loss, and 600K as the IBNR.

2.1.2 Method-X

We could reason that one loss has already reached the limit, and the other one, when multiplied by the LDF will exceed the limit. We remove the two largest losses from the incurred amount and limit their development to the limit. The ultimate values of the $500K & 450K losses will be assumed to be $500K

We would calculate the ultimate loss as follows:
($3M – 450K – 500K) *1.2 + 500K + 500K =
($2.05M)*1.2 + $1M =
$2.46M + $1M =
$3.46M

2.1.3 Method-A vs. Method-X

Method-A gave us an ultimate of $3.6 million whereas Method-X gave us an ultimate of $3.46 million. Consequently, the IBNR from Method-X is $140K lower[3].

Notice that the result from Method-X will always be less than or equal to the result from Method-A. The two will be equal if there are no large losses in the accident year. Sometimes there is pressure for an actuary to produce a lower value of IBNR, and so the second method may be attractive. Nevertheless, as actuaries, we must be careful to use methods that are actuarially sound.

[3] 140K/600K is about 23%, a significant difference in IBNR.

2.2 Background – Notation

Papers written for the CAS are required to use notation consistent with that used in ***The Analysis and Estimation of Loss & ALAE Variability: A Summary Report written*** by the CAS Working Party on Quantifying Variability in Reserve Estimates. This paper uses standard actuarial triangles such as those referred to in the above mentioned report. Some notation follows.

w: The accident year

d: The age of the losses. If the accident year is 2010, then *d*=1 at 12/31/2010 and *d*=2 at 12/31/2011

f(*d*) : Incremental LDF. f(*d*) is applied to a value at age *d* to estimate the value at age *d*+1

F(*d*) : Cumulative LDF. F(*d*) is applied to a value at age *d* to estimate the value at age n. In our examples, n=10, and there is no development after age 9, so F(*d*) estimates the ultimate value of the developing quantity.

$f^T(d)$: true value of f(*d*) for unlimited losses.

$F^T(d)$: true value of F(*d*) for unlimited losses.

Throughout this paper, losses are expressed in thousands (000), or "K" and the retention/limit is $500K.

2.3 Background – Different Sources of LDFs

Suppose that you are given the four triangles below, which are all created with the same underlying data. Triangle "A" contains the aggregated unlimited losses by accident year. Triangle "B" is the same as Triangle "A" except that any occurrences of 500K[4] or more have been limited to 500K (the retention). Triangle "C" is composed only of losses less than 450K (90% of the retention) at the most recent valuation. Triangle "D" is composed only of losses greater than or equal to 450K at the last evaluation, and each of the losses has been limited to 500K. Note that triangle "B" = "C"+"D".

Table 1

A) Unlimited Triangle

	1	2	3	4	5
2009	415	853	1,258	1,654	2,051
2010	180	370	546	717	-
2011	580	1,192	1,758	-	
2012	180	370	-	-	
2013	415	-	-	-	

	1	2	3	4	5
f(d)	2.06	1.48	1.32	1.24	1.00
F(d)	4.94	2.41	1.63	1.24	1.00

B) Limited Triangle 500K per Occ

	1	2	3	4	5
2009	415	839	1,000	1,158	1,316
2010	180	370	546	717	-
2011	580	1,178	1,500	-	
2012	180	370	-	-	
2013	415	-	-	-	

	1	2	3	4	5
f(d)	2.03	1.28	1.21	1.14	1.00
F(d)	3.58	1.76	1.38	1.14	1.00

C) Small Only - Only Losses <= 450K

	1	2	3	4	5
2009	165	339	500	658	816
2010	180	370	546	717	-
2011	180	370	546	-	
2012	180	370	-	-	
2013	415	-	-	-	

	1	2	3	4	5
f(d)	2.06	1.48	1.32	1.24	1.00
F(d)	4.94	2.41	1.63	1.24	1.00

D) Large Only Limited to 500K per Occ

	1	2	3	4	5
2009	250	500	500	500	500
2010	-	-	-	-	-
2011	400	808	955	-	
2012	-	-	-	-	
2013	-	-	-	-	

	1	2	3	4	5
f(d)	2.01	1.11	1.00	1.00	1.00
F(d)	2.24	1.11	1.00	1.00	1.00

The LDFs calculated by the actuary will depend on the triangle used. In some consulting situations, the actuary may only be provided with triangle "B". When this paper refers to "true LDFs" or "true unlimited LDFs" it is referring to LDFs calculated using the unlimited losses, as in triangle "A". When this paper refers to limited LDFs, it is referring to LDFs calculated from a

[4] Actually any occurrences at last evaluation that are 90% of 500K =450K have been limited to 500K. the assumption is that if a loss is 450K at the most recent evaluation, then it will develop to a loss greater than or equal to 500K.

triangle such as "B". A third method, illustrated in the Appendices uses the LDFs from triangles "C" and "D". Within this paper, it is assumed that the "true LDFs" are known and deterministic. In this paper, the universe of examples is created by the author, and in order to simplify the picture of what is happening, the author (me) has assumed[5] that the value of all *unlimited* incurred losses at year 2 is equal to the [unlimited value at year 1] x 2.055 and that the incurred unlimited value at the end of year 3 is equal to the [unlimited value at year 2] x 1.475 etc. etc. This is a very simple model that allows for easy comparison of accuracy of two methods. I do not believe that introducing random fluctuations in the losses would change the result, but it would make the reasoning harder to follow. See the Appendices for some sensitivity testing with regard to changes in LDFs and the ratio of small to large losses. Another author is welcome to explore the effects of random fluctuations in the incurred losses, but in this paper it is assumed that unlimited incurred losses follow the deterministic path described by the LDFs below. The superscript "T" is used to indicate "true unlimited LDFs"

Table 2-True Unlimited LDFs

d	1	2	3	4	5	6	7	8	9	10
$f^T(d)$	2.055	1.475	1.315	1.240	1.200	1.175	1.145	1.125	1.110	1.000
$F^T(d)$	9.964	4.849	3.287	2.500	2.016	1.680	1.430	1.249	1.110	1.000

For every simulated *unlimited* triangle in this paper[6], the calculated LDFs will be f^T and F^T. Note, however, that if random large losses are added to the triangles, and the losses are *limited* to 500K per occurrence, then the limited LDFs will be different for every triangle and dependent on the number, size and accident year of the random large losses. Another way to say this is that changing the large losses in triangle "D" above will change the LDFs calculated from triangle "B" = "C"+ "D".

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[5] In the appendices different assumptions are explored.

[6] In Appendix E, the effects of using different values for f^T and F^T are examined, but in the main part of the paper, only values in Table 2 are used.

2.4 Creating a Simulated Triangle

2.4.1 An Accident Year of Unlimited Occurrences

Suppose we have 14 losses in accident year 2006. One of them has an initial value of 250K, a second has an initial value of 150K and the rest begin at 15K each. The losses would develop as follows. The unlimited development follows Table 2.

Table 3

Year	*d*=1	*d* =2	*d* =3	*d* =4	*d* =5	*d* =6	*d* =7	*d* =8
2006	250	514	758	996	1,236	1,483	1,742	1,995
2006	150	308	455	598	741	890	1,045	1,197
2006	15	31	45	60	74	89	105	120
2006	15	31	45	60	74	89	105	120
2006	15	31	45	60	74	89	105	120
2006	15	31	45	60	74	89	105	120
2006	15	31	45	60	74	89	105	120
2006	15	31	45	60	74	89	105	120
2006	15	31	45	60	74	89	105	120
2006	15	31	45	60	74	89	105	120
2006	15	31	45	60	74	89	105	120
2006	15	31	45	60	74	89	105	120
2006	15	31	45	60	74	89	105	120
2006	15	31	45	60	74	89	105	120
Total	580	1,194	1,753	2,314	2,865	3,441	4,047	4,632

The row in the unlimited triangle would look as follows. If you calculate the incremental LDFs you will see that they match those in Table 2.

Table 4 – Row in an Unlimited Triangle

Year	*d* =1	*d* =2	*d* =3	*d* =4	*d* =5	*d* =6	*d* =7	*d* =8
2006	580	1,194	1,753	2,314	2,865	3,441	4,047	4,632

2.4.2 A Row in a Limited Triangle

The per occurrence values are the same except for the two losses exceeding 500K.

Table 5

Year	*d* =1	*d* =2	*d* =3	*d* =4	*d* =5	*d* =6	*d* =7	*d* =8
2006	250	500	500	500	500	500	500	500
2006	150	308	455	500	500	500	500	500
2006	15	31	45	60	74	89	105	120
2006	15	31	45	60	74	89	105	120
2006	15	31	45	60	74	89	105	120
2006	15	31	45	60	74	89	105	120
2006	15	31	45	60	74	89	105	120
2006	15	31	45	60	74	89	105	120
2006	15	31	45	60	74	89	105	120
2006	15	31	45	60	74	89	105	120
2006	15	31	45	60	74	89	105	120
2006	15	31	45	60	74	89	105	120
2006	15	31	45	60	74	89	105	120
2006	15	31	45	60	74	89	105	120
Total	580	1,180	1,495	1,720	1,888	2,068	2,260	2,440

The row in the *limited* triangle would look as follows. If you calculate the incremental LDFs you will see that they are lower than those in Table 2. [7]An examples of one iteration of *limited* occurrences is given in Appendix A[8]

Table 6-Row in a Limited Triangle

Year	*d* =1	*d* =2	*d* =3	*d* =4	*d* =5	*d* =6	*d* =7	*d* =8
2006	580	1180	1495	1720	1888	2068	2260	2440

2.4.3 Actual Values of Ultimate Losses

Since unlimited loss development is known exactly, we can calculate the exact ultimate values of each loss on both a limited and unlimited basis.

Table 7 – Distribution of Random Losses

Value of Occurrence at *d* =1	$F^T(1)$	Ultimate Unlimited Loss	Ultimate Limited Loss
15	9.964	149.46	149.46
150	9.964	1494.60	500.00
250	9.964	2491.00	500.00

[7] The row below in Table 6 is similar to the 2006 row in Table 9. The only reason for differences is rounding.

[8] Applicable mathematical formulae are included in Appendix C.

The fact that the model is set up so that the "true" answer is known means that we can evaluate different methods to see which method is closer to the true answer. The occurrences are aggregated to get results by accident year.

2.5 Methods

First, initial occurrence values are selected for each accident year, and unlimited occurrences are developed using $f^T(d)$, i.e. the "true" incremental LDFs. Then, the occurrences are limited to 500K, and a triangle is created by aggregating the occurrences by accident year.

1) Limited LDFs, $f(d)$ & $F(d)$, are calculated from the limited triangle using an all-year weighted average.
2) The loss development factors from 1) are applied to
   a. All the limited losses. This will be referred to as "Method-A".
   b. Incurred losses excluding the losses within 90% of the limit. After application of the LDF, the large losses are added back in at full limits. This will be referred to as "Method-X"
3) The methods above are investigated for accuracy, bias and adequacy.

The number of occurrences in each accident year stays the same from trial to trial. Also in each accident year, most occurrences are static at 15K, but there are two random losses. The probability distribution of the two random occurrences is given below. In each accident year, there is a possibility that zero, one or two occurrences will have ultimate values greater than or equal to the retention. The incurred value of each individual claim at age 1 is chosen from the values of $15K, $150K and $250K. Values of $15K at 1 year do not reach the retention limit at maturity. Initial values of 150K and 250K both exceed the 500K limit after some development. Table 5 shows one accident year of simulated losses. A full set of simulated losses from one iteration is shown in Appendix A.

Table 8 – Random Losses for Each Accident Year

Size of Random Occurrences at d=1	Probability
15K & 15K	64%
250K & 15K	16%
150K & 15K	16%
250K&150K	2%
250K & 250K	1%
150K & 150K	1%

During each iteration, a limited triangle is simulated and set of limited LDFs is derived from this triangle. Below is one simulated triangle with losses limited to 500K and the associated all-year weighted incurred LDFs. Note that the LDFs calculated from the limited incurred triangle are smaller than f^T and F^T. Since we know the true development factors, we can calculate the *actual* ultimate losses, and can compare methods for accuracy. We will first look at a single iteration.

2.5.1 Limited Triangle from One Iteration

Table 9

	1	2	3	4	5	6	7	8	9	10
2004	180	370	546	717	890	1,068	1,254	1,436	1,616	1,794
2005	180	370	546	717	890	1,068	1,254	1,436	1,616	-
2006	580	1,178	1,500	1,717	1,890	2,068	2,254	2,436	-	
2007	180	370	546	717	890	1,068	1,254	-		
2008	415	839	1,000	1,158	1,316	1,479	-			
2009	180	370	546	717	890	-	-			
2010	225	462	682	897	-	-	-			
2011	400	808	955	-						
2012	180	370	-	-						
2013	285	-	-	-						

2.5.2 Limited LDFs from One Iteration

Table 10

	1	2	3	4	5	6	7	8	9	10
f(d)	2.04	1.33	1.24	1.18	1.15	1.14	1.11	1.13	1.11	1.00
F(d)	7.19	3.53	2.66	2.15	1.83	1.59	1.39	1.25	1.11	1.00

2.5.3 Results from the Application of Method-A

Table 11

Accident Year	**Age - d**	**Incurred $(000)**	**F(d)**	**Method A Estimate $(000)**	**True Ultimate $(000)**	**Method A IBNR $(000)**	**True IBNR $(000)**	**Error $(000)**	**Error as a % of True IBNR**
		(a)	**(b)**	**(c)**	**(d)**	**(e)**	**(f)**	**(g)**	**(h)**
		modeled	**modeled**	**=(a)*(b)**		**=(c) -(a)**	**=(d)-(a)**	**=(f)-(e)**	**=(g)/(f)**
2004	10	1,794	1.00	1,794	1,794	-	-	-	-
2005	9	1,616	1.11	1,794	1,794	178	178	-	0%
2006	8	2,436	1.25	3,042	2,794	606	357	248.7	70%
2007	7	1,254	1.39	1,746	1,794	491	539	(47.7)	-9%
2008	6	1,479	1.59	2,350	2,144	871	666	205.6	31%
2009	5	890	1.83	1,624	1,794	735	904	(169.2)	-19%
2010	4	897	2.15	1,928	2,242	1,031	1,345	(314.1)	-23%
2011	3	955	2.66	2,541	1,995	1,586	1,040	546.0	53%
2012	2	370	3.53	1,305	1,794	935	1,424	(488.7)	-34%
2013	1	285	7.19	2,050	1,845	1,765	1,560	204.4	13%
Total		**11,975**		**20,173**	**19,988**	**8,198**	**8,013**	**185.1**	**2%**
2004-2011		**11,320**		**16,818**	**16,349**	**5,498**	**5,029**	**469.3**	**9%**

2.5.4 Results from the Application of Method-X

Table 12- Application of Method-X

Accident Year	Age - d	F(d)	Incurred $(000)	Large Losses	Inc X Known Large Losses	Method X Estimate $(000)	True Ultimate $(000)	Method X IBNR $(000)	True IBNR $(000)	Error $(000)	Error as a % of True IBNR
		(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)	(j)
		modeled	modeled	modeled	=(b)-(c)	=(a)*(d) + (c)		=(e) -(b)	=(f)-(b)	=(g)-(h)	=(i)/(h)
2004	10	1.00	1,794	-	1,794	1,794	1,794	-	-	-	$ -
2005	9	1.11	1,616	-	1,616	1,794	1,794	178	178	-	0%
2006	8	1.25	2,436	1,000	1,436	2,794	2,794	357	357	(0)	0%
2007	7	1.39	1,254	-	1,254	1,746	1,794	491	539	(48)	-9%
2008	6	1.59	1,479	500	979	2,055	2,144	577	666	(89)	-13%
2009	5	1.83	890	-	890	1,624	1,794	735	904	(169)	-19%
2010	4	2.15	897	-	897	1,928	2,242	1,031	1,345	(314)	-23%
2011	3	2.66	955	500	455	1,710	1,995	755	1,040	(285)	-27%
2012	2	3.53	370	-	370	1,305	1,794	935	1,424	(489)	-34%
2013	1	7.19	285	-	285	2,050	1,845	1,765	1,560	204	13%
Total			11,975	2,000		**18,799**	**19,988**	**6,824**	**8,013**	**(1,189)**	**-15%**
2004-2011			**11,320**	**2,000**		**15,444**	**16,349**	**4,124**	**5,029**	**(905)**	**-18%**

2.5.5 Comments on Results

Note that the results from Method-A (i.e. applying the limited LDF to all losses) is more accurate. Note also that the result of Method-A is conservative and the result from Method-X is deficient.

2.6 Simulation Results 10,000 Trials

A model was created that simulates 10 years of loss data. The same techniques used in the prior sub-sections were applied. Large losses are randomly allocated to the accident years. For each year, there is a possibility of between zero and two large losses. Table 8 provides the probability of the incurred losses by accident year and severity at d=1. The probability of large losses in any one year is independent of the number and size of losses in any other year.

Method-A :(All Losses) The LDF was applied to all limited losses regardless of size.

Method-X :(All losses excluding max limit losses.) All losses within 90% of the retention were removed from the incurred losses prior to application of the LDF. The LDF was then applied to all remaining losses. After application of the LDF, the large losses were added back in at the max retention.

In each case, the percentage error between the true ultimate losses and the calculated ultimate losses was found.

2.6.1 Comparison of Methods: Mean, Bias, Adequacy

The results of 10,000 simulations are shown below. A negative error indicates an aggressive (low) estimate, whereas a positive error indicates a conservative (high) estimate.

Table 13

		Error as a Percentage of IBNR						Error as % of Ult
	Method	**10th Percentile**	**25th Percentile**	**Mean Error**	**75th Percentile**	**90th Percentile**	**Std Dev**	**Mean**
All Years	A	-19%	**-10%**	**3.4%**	15%	28%	19%	1.35%
2004-2011	A	-20%	**-11%**	**1.9%**	14%	25%	17%	0.54%
All Years	X	-34%	**-27%**	**-20%**	-12%	-5%	11%	-7.90%
2004-2011	X	-32%	-25%	-18%	-9%	-3%	11%	-5.54%

Note that both methods are biased, but Method-X more so. Method-X is so biased that 2004-2011 estimates are less than or equal to the true value in 99% of the simulations.

2.6.2 Comparison of Methods: Distance from the True Ultimate

The Table below calculates the absolute value of the error from Method-X minus the absolute value of the error from Method-A. The fact that the mean is positive indicates that the result from Method-A is expected to be closer than that from Method-X. Note that the difference is more pronounced if the methods exclude the two most recent years.

Table 14 - Difference Between Absolute Errors
Abs (Error Method-X) – Abs (Error Method A)

	10th Percentile	**Mean**	**90th Percentile**	**Std Dev**
All Years	-19%	1.7%	18%	15%
2004-2011	-18%	4.0%	20%	15%

2.6.3 Comparison of Methods: A Subjective Measure

In many instances an overestimate of reserves is preferable to an underestimate. An underestimate could lead to underpricing, (negative income) or unfavorable reserve adjustments in later years. If "conservative" error is preferable to "aggressive" error, the percentages in the above table are understated. For example, if conservative error is determined to be only 70% as "wrong" as a low estimate, then a new error term "Subjective Error" could be defined where

EA = (total error for 1 iteration / True IBNR for 1 iteration) for Method-A

EX = (total error for 1 iteration / True IBNR for 1 iteration) for Method-X

Subjective_Error-A = MAX [70%(EA), -EA]

Subjective_Error-X = MAX [70%(EX), -EX], and the subjective superiority of Method-A could be defined by: (Subjective_ Error-X – Subjective_ Error-A)

Note that based on the above definitions, the difference between the two methods is positive if Method-A is closer to the true answer and negative if Method-X is closer to the true answer, where the "closeness" is adjusted for conservative estimates. If reserve deficiency is less desirable than reserve redundancy by a ratio of 10-to-7 then the following results occur.

Table 15 Difference Between Subjective Errors
(Subjective_Error-X) – (Subjective_Error-A)

	10th Percentile	Mean	90th Percentile	Std Dev
All Years	-13%	4%	19%	12%
2004-2011	-10%	6%	21%	12%

The results above show that if conservative error is favored then Method-A on average is more accurate by about 6% for years 2004 through 2011 and about 4% for all years combined. Additional simulation results are shown in Appendix B.

3. RESULTS AND DISCUSSION

Table 16 – Sample Incurred Losses

Year	Unlimited Incurred (000)	Limited Incurred (000)	#Losses 500K or More	Sum of Max Limit Losses (000)
2004	897	897	-	0
2005	936	936	-	0
2006	1,922	922	2	1000
2007	1,701	1,701	-	0
2008	3,296	2,296	2	1000
2009	769	769	-	0
2010	1,603	1,103	1	500
2011	3,346	2,400	2	954
2012	244	244	-	0
2013	257	257	-	0

Let's suppose that you are the consulting actuary for a company with a self-insured retention of $500K per claim. You have been given the above incurred data as of 12/31/2013. Also, the company has provided you with an incurred triangle with losses limited to 500K.

3.1 The Only Triangle Available is Limited to 500K

From the triangle with limited losses you obtain the following factors.

Table 17

d	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
f(*d*)	4.03	1.23	1.32	1.10	1.05	1.05	1.05	1.03	1.00
F(*d*)	8.61	2.13	1.74	1.31	1.19	1.14	1.08	1.03	1.00

You note that there 7 maximum limit losses that can not develop beyond 500K

What should you do? Based on the results of this paper, the most accurate method is to apply the LDFs obtained from the limited triangle to *all* losses, regardless of whether or not each individual loss has reached the maximum limit.

Intuitively this makes sense. If a factor is developed using all the truncated losses, then the factor should be applied to all the truncated losses so that the factor is consistent with the underlying data. It doesn't make sense to apply a factor developed with one type of data to dissimilar data where the differences are known and avoidable.

3.2 If Detailed Data is Available

If you are able to obtain detailed data, and create a triangle "S" that contains only the losses less then 90% of the retention, then the factors from "S" could be applied to the smaller losses. You could create two triangles and sets of LDFs: one for small losses and the other for large losses. It is not the intent of this paper to prove that separation of large and small losses is preferable. However, for the examples and simulations in this paper, separation of large and small losses is more accurate on average. See Appendix D for an example.

4. CONCLUSIONS

It is not actuarially sound to remove truncated/limited losses from incurred and/or paid results if limited loss development factors are applied. The loss development factors should be consistent with the losses to which they are applied to the extent possible.

Acknowledgment

Thank you to Patrick Glenn, Robert Kell, Jennifer Wu and John Xu for their comments and edits.

Appendix A

A single iteration of the simulated occurrences is listed below.

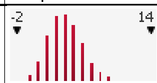
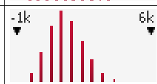
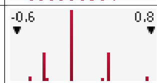
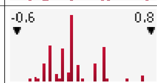
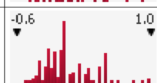
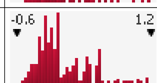
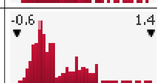
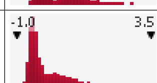
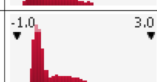
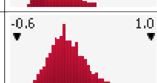
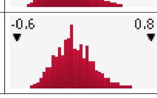
Year	1	2	3	4	5	6	7	8	9	10	True Ult	Large Amt
2004	15	31	45	60	74	89	105	120	135	149	149	0
2004	15	31	45	60	74	89	105	120	135	149	149	0
2004	15	31	45	60	74	89	105	120	135	149	149	0
2004	15	31	45	60	74	89	105	120	135	149	149	0
2004	15	31	45	60	74	89	105	120	135	149	149	0
2004	15	31	45	60	74	89	105	120	135	149	149	0
2004	15	31	45	60	74	89	105	120	135	149	149	0
2004	15	31	45	60	74	89	105	120	135	149	149	0
2004	15	31	45	60	74	89	105	120	135	149	149	0
2004	15	31	45	60	74	89	105	120	135	149	149	0
2004	15	31	45	60	74	89	105	120	135	149	149	0
2004	15	31	45	60	74	89	105	120	135	149	149	0
2005	15	31	45	60	74	89	105	120	135	-	149	0
2005	15	31	45	60	74	89	105	120	135		149	0
2005	15	31	45	60	74	89	105	120	135		149	0
2005	15	31	45	60	74	89	105	120	135		149	0
2005	15	31	45	60	74	89	105	120	135		149	0
2005	15	31	45	60	74	89	105	120	135		149	0
2005	15	31	45	60	74	89	105	120	135		149	0
2005	15	31	45	60	74	89	105	120	135		149	0
2005	15	31	45	60	74	89	105	120	135		149	0
2005	15	31	45	60	74	89	105	120	135		149	0
2005	15	31	45	60	74	89	105	120	135		149	0
2005	15	31	45	60	74	89	105	120	135		149	0
2006	250	500	500	500	500	500	500	500			500	500
2006	150	308	455	500	500	500	500	500			500	500
2006	15	31	45	60	74	89	105	120			149	0
2006	15	31	45	60	74	89	105	120			149	0
2006	15	31	45	60	74	89	105	120			149	0
2006	15	31	45	60	74	89	105	120			149	0
2006	15	31	45	60	74	89	105	120			149	0
2006	15	31	45	60	74	89	105	120			149	0
2006	15	31	45	60	74	89	105	120			149	0
2006	15	31	45	60	74	89	105	120			149	0
2006	15	31	45	60	74	89	105	120			149	0
2006	15	31	45	60	74	89	105	120			149	0
2006	15	31	45	60	74	89	105	120			149	0
2006	15	31	45	60	74	89	105	120			149	0
2007	15	31	45	60	74	89	105	-			149	0
2007	15	31	45	60	74	89	105	-			149	0
2007	15	31	45	60	74	89	105				149	0
2007	15	31	45	60	74	89	105				149	0
2007	15	31	45	60	74	89	105				149	0
2007	15	31	45	60	74	89	105				149	0
2007	15	31	45	60	74	89	105				149	0
2007	15	31	45	60	74	89	105				149	0
2007	15	31	45	60	74	89	105				149	0
2007	15	31	45	60	74	89	105				149	0
2007	15	31	45	60	74	89	105				149	0
2007	15	31	45	60	74	89	105				149	0
2008	250	500	500	500	500	500	-				500	500
2008	15	31	45	60	74	89	-				149	0
2008	15	31	45	60	74	89					149	0
2008	15	31	45	60	74	89					149	0
2008	15	31	45	60	74	89					149	0
2008	15	31	45	60	74	89					149	0
2008	15	31	45	60	74	89					149	0
2008	15	31	45	60	74	89					149	0
2008	15	31	45	60	74	89					149	0
2008	15	31	45	60	74	89					149	0
2008	15	31	45	60	74	89					149	0
2008	15	31	45	60	74	89					149	0

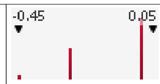
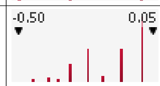
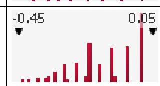
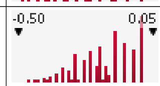
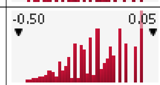
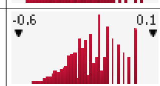
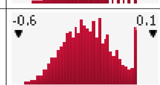
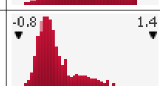
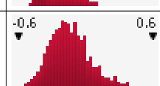
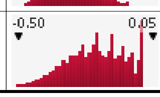
Year	1	2	3	4	5	6	7	8	9	10	True	Large
2009	15	31	45	60	74	-					149	0
2009	15	31	45	60	74	-					149	0
2009	15	31	45	60	74						149	0
2009	15	31	45	60	74						149	0
2009	15	31	45	60	74						149	0
2009	15	31	45	60	74						149	0
2009	15	31	45	60	74						149	0
2009	15	31	45	60	74						149	0
2009	15	31	45	60	74						149	0
2009	15	31	45	60	74						149	0
2009	15	31	45	60	74						149	0
2009	15	31	45	60	74						149	0
2010	15	31	45	60	-						149	0
2010	15	31	45	60	-						149	0
2010	15	31	45	60							149	0
2010	15	31	45	60							149	0
2010	15	31	45	60							149	0
2010	15	31	45	60							149	0
2010	15	31	45	60							149	0
2010	15	31	45	60							149	0
2010	15	31	45	60							149	0
2010	15	31	45	60							149	0
2010	15	31	45	60							149	0
2010	15	31	45	60							149	0
2010	15	31	45	60							149	0
2010	15	31	45	60							149	0
2011	250	500	500	-							500	500
2011	15	31	45	-							149	0
2011	15	31	45	-							149	0
2011	15	31	45	-							149	0
2011	15	31	45	-							149	0
2011	15	31	45	-							149	0
2011	15	31	45	-							149	0
2011	15	31	45	-							149	0
2011	15	31	45	-							149	0
2011	15	31	45								149	0
2011	15	31	45								149	0
2012	15	31	-								149	0
2012	15	31	-								149	0
2012	15	31	-								149	0
2012	15	31	-								149	0
2012	15	31	-								149	0
2012	15	31	-								149	0
2012	15	31	-								149	0
2012	15	31	-								149	0
2012	15	31	-								149	0
2012	15	31	-								149	0
2012	15	31	-								149	0
2012	15	31	-								149	0
2013	150	-									500	0
2013	15	-									149	0
2013	15	-									149	0
2013	15	-									149	0
2013	15	-									149	0
2013	15	-									149	0
2013	15	-									149	0
2013	15	-									149	0
2013	15	-									149	0
2013	15	-									149	0

Appendix B

Selected results of a simulation are shown below. Rows labeled Meth_X refer to the results of applying Method-X, Rows labeled Meth_A refer to the results of applying Method-A. Rows labeled "Total" refer to all accident years combined, and rows labeled"Total_04_11" refer to results from combining all accident years except for the two most recent.

Table 18

Name	Graph	Min	Mean	Max	5%	95%
Number of Large Losses		-	4	13	1	7
Amount of Losses 450K or more on the evaluation date.		-	1,691	5,455	500	3,000
Meth A-2005-error		-43%	3%	74%	-25%	34%
Meth A-2006-error		-46%	0%	70%	-28%	32%
Meth A-2007-error		-44%	3%	96%	-30%	43%
Meth A-2008-error		-46%	3%	112%	-31%	51%
Meth A-2009-error		-48%	4%	135%	-32%	61%
Meth A-2010-error		-53%	-1%	129%	-37%	60%
Meth A-2011-error		-57%	8%	244%	-40%	93%
Meth A-2012-error		-61%	5%	324%	-43%	112%
Meth A-2013-error		-54%	6%	253%	-39%	104%
Meth A-Total-error		-44%	3%	86%	-25%	37%
Meth A-2004-2011-error		-43%	2%	76%	-25%	33%

Meth X-2005-error		-43%	-10%	0%	-25%	0%
Meth X-2006-error		-46%	-11%	0%	-33%	0%
Meth X-2007-error		-44%	-13%	0%	-32%	0%
Meth X-2008-error		-46%	-14%	0%	-33%	0%
Meth X-2009-error		-48%	-17%	0%	-36%	0%
Meth X-2010-error		-53%	-20%	0%	-39%	0%
Meth X-2011-error		-57%	-23%	0%	-42%	-2%
Meth X-2012-error		-61%	-16%	122%	-44%	39%
Meth X-2013-error		-54%	6%	253%	-39%	104%
Meth X-Total-error		-52%	-13%	49%	-35%	14%
Meth X-2004-2011-error		-50%	-18%	0%	-36%	-1%

Appendix C– Mathematical Formulae

Much of the following is taken from ***The Analysis and Estimation of Loss & ALAE Variability: A Summary Report*** written by the CAS Working Party on Quantifying Variability in Reserve Estimates.

The row dimension is the annual period by which the loss information is subtotaled, most commonly an accident year or policy year. For each accident period, w, the (w, d) element of the array is the total of the loss information as of development age d. Here the development age is expressed as the number of time periods after the accident or policy year. For example, the loss statistic for accident year 2 as of the end of calendar year 4 has development age 3 years.

For this discussion, we assume that the loss information available is an "upper triangular" subset of the two-dimensional array for rows $w = 1,2,\ldots,n$. For each row, w, the information is available for development ages 1 through $n - w+1$. If we think of year as the latest accounting year for which loss information is available, the triangle represents the loss information as of accounting dates 1 through n. The "diagonal" for which $w + d = k$, a constant, represents the loss information for each accident period w as of accounting year k.

The creation of simulated losses within this paper assumes that unlimited loss development is known exactly. An initial loss is chosen at d=1 year. Losses at subsequent ages are found by multiplying by the incremental development factors. If the loss exceeds the retention at any age then, then the limited loss is set at the retention. The following table provides the mathematical formulae for calculating simulated limited losses.

$$c(w, i, d+1) = \mathrm{MIN}[c(w, i, d) * f^{T}(d), \text{retention}] \quad \text{(E.1)}$$

$$U(w, i) = \mathrm{MIN}[c(w, i, 1) * F^{T}(1), \text{retention}] \quad \text{(E.2)}$$

w: The accident year

d: The age of the losses. If the accident year is 1/1/2010 to 12/31/2010, then d=1 at 12/31/2010 and d=2 at 12/31/2011

i: denotes an occurrence within an accident year.

$c(w, d)$: cumulative loss from accident (or policy) year w as of age d.

$c(w, i, d)$: cumulative loss from the i^{th} occurrence in accident (or policy) year w as of age d .

$$c(w, d) = \sum_{i} c(w, i, d).$$

$c(w, i, \infty) = U(w, i)$**:** ultimate loss from the i^{th} occurrence in accident year w.

f(*d*): factor applied to c(*w*, *i*, *d*) or *c*(*w*, *d*) to estimate c(*w*, *i*, *d*+1) or c(*w*, *d* +1) respectively.

F(*d*) : factor applied to c(*w*, *i*, *d*) or *c*(*w*, *d*) to estimate c(*w*, *i*, *n*) or c(*w*, *n*) respectively.

$f^T(d)$: true value of f(*d*) for unlimited losses.

$F^T(d)$: true value of F(*d*) for unlimited losses.

U(*w*, ***i*)*:*** ultimate loss for the i^{th} occurrence in accident year *w*.

$U(w) = \sum_i U(w, i)$: The ultimate loss for accident year *w*.

Throughout this paper, losses are expressed in thousands (000), or "K", the retention = $500K. Also, *w* is the accident year, *d* is the age of the accident year, and *i* refers to a particular occurrence within the accident year. The values of F(*d*) and f(*d*) are cumulative and incremental LDFs respectively, and will vary depending on the triangle and methods used in their calculation. The true LDFs, $f^T(d)$ and $F^T(d)$ are defined by the following table (rounded). For more information on notation, please see Appendix C.

Table 19

d	1	2	3	4	5	6	7	8	9	10
$f^T(d)$	2.055	1.475	1.315	1.240	1.200	1.175	1.145	1.125	1.110	1.000
$F^T(d)$	9.964	4.849	3.287	2.500	2.016	1.680	1.430	1.249	1.110	1.000

Here are some examples:

c(2010, 3, 1) = 15 means that the 3rd occurrence in accident year 2010 has a value of 15K at an age of 1.

c(2010, 3, 2) = MIN[c(2010, 3, 1) * f^T (1), 500] =15*2.055 = 30.825 is the incurred value at age 2 for the 3rd occurrence

c(2010, 3, 3) = MIN[c(2010, 3, 2)*f^T (2), 500] = 30.825*1.475 ≈ 45.467 , the incurred value at age 3.

U(2010, 3) = c(2010, 3, 1)*F^T (1) = 15*9.964≈ 148.466. This is the ultimat e value of the third incurred loss in accident year 2010. Note that none of the incurred values for the 3rd occurrence in accident year 2010 will ever exceed the retention.

c(2011, 1, 1) = 250 means that the first occurrence in accident year 2011 has a value of 250K at age *d*=1.

c(2011, 1, 2) = MIN[c(2011, 1, 1)*f^T (1), 500] = MIN[250*2.055, 500] = 500.

Furthermore, f(2011, 1, 3)=500 and U(2011, 1) = 500 because once an incurred loss hits the retention it stays there for subsequent ages.

Examples of these calculations are included implicitly in Appendix A[9]

c(*w*,*d*) is the value of the incurred triangle in the row corresponding to accident year *w*, and column *d* where c(*w*,*d*) = $\sum_i$ c(*w*,*i*,*d*) = the sum of all occurrences in accident year *w* at age *d*. See Appendix A for some actual simulated liabilities.

[9] The mathematical formulae make the process look more complicated than it really is. Imagine a natural method for developing losses from unlimited age-to-age factors, then limiting the losses by occurrence limits. This is what the formulae are doing.

Appendix D – Separate LDFs for large and Small Losses

The following triangle and associated development factors result from only including losses less than 450K (90% of 500K) at the time of evaluation.

Table 20- Triangle of Small Losses

Year	1	2	3	4	5	6	7	8	9	10
2004	180	370	546	717	890	1,068	1,254	1,436	1,616	1,794
2005	180	370	546	717	890	1,068	1,254	1,436	1,616	-
2006	180	370	546	717	890	1,068	1,254	1,436	-	-
2007	180	370	546	717	890	1,068	1,254	-	-	-
2008	165	339	500	658	816	979	-		-	-
2009	180	370	546	717	890	-			-	-
2010	225	462	682	897	-				-	-
2011	150	308	455	-	-	-	-	-	-	-
2012	180	370	-							
2013	285	-								

Table 21-LDFs for Small Losses

d	1	2	3	4	5	6	7	8	9
f(*d*)	2.06	1.48	1.32	1.24	1.20	1.18	1.15	1.13	1.11
F(*d*)	9.96	4.85	3.29	2.50	2.02	1.68	1.43	1.25	1.11

Note that the removal of the large losses allowed for calculation of the true LDFs as shown in Table 2, and that these LDFs are significantly higher than those calculated with truncated losses.

The next triangle and associated development factors result from only including losses less than 450K *or more* at the time of evaluation.

Table 22 - Triangle of Large Losses

Year	1	2	3	4	5	6	7	8	9	10
2004									-	
2005	-	-	-	-	-	-	-	-	-	-
2006	400	808	955	1000	1000	1000	1000	1000	-	-
2007	-	-	-	-	-	-	-	-	-	-
2008	250	500	500	500	500	500	-	-	-	-
2009						-	-	-	-	-
2010	-	-	-	-	-	-	-	-	-	-
2011	250	500	500	-	-	-	-	-	-	-
2012	-	-	-	-	-	-	-	-	-	
2013	-	-	-	-	-	-	-	-	-	

Table 23- LDFs for Large Losses

d	1	2	3	4	5	6	7
f(*d*)	2.01	1.08	1.03	1.00	1.00	1.00	1.00
F(*d*)	2.24	1.11	1.03	1.00	1.00	1.00	1.00

We will now apply these LDFs separately to large and small losses.

Table 24-Chain Ladder for Large and Small Losses Separately

Accident Year	Age - *d*	Small Losses	F(*d*)	Ultimate Part-1		Large Losses	F(*d*)	Ultimate Part-2
		(a)	(b)	'(c)		(d)	(e)	(f)
		Table 17	Table 18	=(a)*(b)		Table 19	Table 20	=(d)*(e)
2004	10	1,794	1.00	1,794		-	1.00	-
2005	9	1,616	1.11	1,794		-	1.00	-
2006	8	1,436	1.25	1,794		1,000	1.00	1,000
2007	7	1,254	1.43	1,794		-	1.00	-
2008	6	979	1.68	1,644		500	1.00	500
2009	5	890	2.02	1,794		-	1.00	-
2010	4	897	2.50	2,242		-	1.00	-
2011	3	455	3.29	1,495		500	1.03	516
2012	2	370	4.85	1,794		-	1.11	-
2013	1	285	9.96	2,840		-	2.24	-
Total		**9,975**		**18,982**		**2,000**	**11**	**2,016**
2004-2011		**9,320**		**14,349**		**2,000**	**8**	**2,016**

Table 25-Error as a Percentage of IBNR

Accident Year	Small+Large Ult	True Ult	Difference	Error as % of IBNR
	(g)	(h)	(i)	(c)
	(c)+(f)	calc		(i)/[(h)-(a)-(d)]
2004	1,794	1,794	-	0%
2005	1,794	1,794	-	0%
2006	2,794	2,794	-	0%
2007	1,794	1,794	-	0%
2008	2,144	2,144	-	0%
2009	1,794	1,794	-	0%
2010	2,242	2,242	-	0%
2011	2,010	1,995	16	1%
2012	1,794	1,794	-	0%
2013	2,840	1,845	995	64%
Total	20,998	19,988	1,010	13%
2004-2011		**16,349**	**16**	**0%**

Note that the result is nearly perfect. The only year that is off significantly is 2013. That year is "off" because there is a loss in 2013 that will reach over 500K, but has not yet been detected as a large loss. This paper is not designed to explore methods of separating large and small losses, or to prove that separating the two is always better.

Appendix E– Sensitivity

This appendix explores the effects of altering the LDFs, and altering the percentage of large (limited) losses in relationship to ultimate losses. For the tables below only 5000 simulations were used in each of the nine scenarios. In this appendix, a slightly different method is used for identifying large losses. If the value at time *d* multiplied by F(*d*) is larger than 500K, then in Method-X, the loss for that occurrence is limited to 500K. The ratio of large to small large losses was changed by altering the value of the static small losses. It can be seen that the magnitude of the errors changes, but the fact that Method-X is biased toward low estimates is unchanged.

Table 26- All Years Combined – Sensitivity of Mean Error to LDF and Percentage of Large Losses

		Highest LDF		High LDF		Moderate LDF	
Ratio of Large Losses to Total Losses - Ultimate Limited Basis	**Method**	**Mean Error as % of IBNR**	**Mean Error as % of Ultimate**	**Mean Error as % of IBNR**	**Mean Error as % of Ultimate**	**Mean Error as % of IBNR**	**Mean Error as % of Ultimate**
15%	A	7%	3%	4%	1%	2%	0%
	X	-33%	-12%	-23%	-6%	-16%	-3%
10%	A	4%	1%	2%	1%	1%	0%
	X	-25%	-10%	-16%	-5%	-12%	-2%
5%	A	1%	1%	1%	0%	0%	0%
	X	-14%	-6%	-9%	-3%	-6%	-1%

Table 27- 2004-2011 – Sensitivity of Mean Error to LDF and Percentage of Large Losses

		Highest LDF		High LDF		Moderate LDF	
Ratio of Large Losses to Total Losses - Ultimate Limited Basis	**Method**	Mean Error as % of IBNR	Mean Error as % of Ultimate	Mean Error as % of IBNR	Mean Error as % of Ultimate	Mean Error as % of IBNR	Mean Error as % of Ultimate
15%	**A**	5%	1%	3%	1%	2%	0%
	X	-26%	-7%	-20%	-4%	-18%	-2%
10%	**A**	2%	1%	1%	0%	1%	0%
	X	-18%	-6%	-14%	-3%	-12%	-2%
5%	**A**	1%	0%	0%	0%	0%	0%
	X	-10%	-3%	-8%	-2%	-6%	-1%

The LDFs used in the sensitivity analysis are as follows.

Table 28 – Highest LDFs

d	1	2	3	4	5	6	7	8	9	10
f(*d*)	2.055	1.475	1.315	1.240	1.200	1.175	1.145	1.125	1.110	1.000
F(*d*)	9.964	4.849	3.287	2.500	2.016	1.680	1.430	1.249	1.110	1.000

Table 29 – High LDFs

d	1	2	3	4	5	6	7	8	9	10
f(*d*)	1.541	1.263	1.179	1.138	1.116	1.102	1.085	1.073	1.065	1.000
F(*d*)	3.973	2.579	2.042	1.733	1.523	1.365	1.239	1.143	1.065	1.000

Table 30 – Moderate LDFs

d	1	2	3	4	5	6	7	8	9	10
f(*d*)	1.296	1.150	1.104	1.081	1.068	1.060	1.050	1.043	1.038	1.000
F(*d*)	2.289	1.766	1.536	1.391	1.287	1.205	1.137	1.083	1.038	1.000

5. REFERENCES

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Calculations performed using @RISK Standard version 5.0, from Palisade Corporation, Ithaca, NY, U.S.A.

Abbreviations and notations

K, one thousand

LDF, loss development factor

w, accident year

F(d) cumulative LDF applicable to accident year at age *d*

(000) number in thousands

CL, Chain Ladder Method

d, delay or age of accident year

f(d) incremental LDF applicable to accident year at age *d*

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